Acta Cryst. (1999). A**55**, 746–754

Quasiperiodic packings of fibres with icosahedral symmetry

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(*Received 4 November 1998; accepted 21 January 1999*)

Abstract

Different icosahedral packings of fibres have been experimentally realized. A packing construction with straight fibres of the same circular cross section, only parallel to fivefold icosahedral axes and respecting the closest packing condition, is reported. Its characteristics of point-group symmetry and related two-dimensional tilings are analysed. But for determining unambiguously all the fibre positions it appears that a mathematical construction has to be made from the cut and projection of a five-dimensional space. Through such a method, the volume fraction of fibrous reinforcement in a composite material can be calculated. The related two-dimensional tiling can be proved to be different from a Penrose tiling. Finally, the characteristics of other icosahedral packings where fibres are parallel to threefold axes or to both threefold and fivefold axes are briefly discussed and a few further experiments on their elasticity properties and photonic band-gap structure are suggested.

1. Introduction

In a recent article, Parkhouse & Kelly (1998) have explored the feasibility of designing a fibrous composite that is both elastically isotropic and contains an appreciable volume fraction of reinforcement. But from their investigations on both periodic and quasiperiodic packings, they conclude that it is not possible to obtain this for a regular array of bars of the same cross section. Their result appears to be in agreement with that of Christensen (1987) showing that a 'six-dimensional' array (*i.e.* an array with six equivalent directions) could be made elastically isotropic but that for infinitely long fibres the attainable volume fraction tends to zero. It should be noted that Parkhouse & Kelly also propose a method for constructing a quasiperiodic packing of icosahedral symmetry with bars of four different cross sections.

Our own interest in quasiperiodic fibre packings in three dimensions derives from our previous studies, over a long period, on the structures of quasicrystalline intermetallic compounds discovered by Shechtman *et al.* (1984). We have presented at a recent conference on quasicrystals (Audier & Duneau, 1998) models of icosahedral packing of fibres of the same cross section. Photographs of two of these models are shown in Fig. 1. One is made of Plexiglas fibres aligned parallel to the six fivefold axes of the regular icosahedron and the other is made of wood fibres aligned parallel to the six fivefold and ten threefold axes of the regular icosahedron. As our studies show that different quasiperiodic packings in three dimensions can be infinite, we were trying to express a few general quasiperiodic packing rules when the article of Parkhouse & Kelly (1998) came to our notice. First, we convinced ourselves that the results of these authors really are different from ours. Then we tried to understand on what basis and for which particular case they have established their conclusions.

After a brief recapitulation of the two types of icosahedral symmetry, we report in the present article a way to construct the tightest packing of fibres of the same circular cross section parallel to the six fivefold axes of the regular icosahedron. The characteristics of such a packing are then analysed, in particular those of its two-dimensional projection along a fivefold axis, with respect to those of a Penrose pattern. A few demonstrations are given, based on a hyperspace description, and related to the closest-packing condition and to density calculation. Finally, we discuss the possibility of realizing other packings where fibres are parallel to fivefold and threefold axes of the icosahedron.

2. Resumé of the icosahedral point groups I_h and I

Such a recapitulation might be useful as the mechanical properties of an icosahedral packing of fibres must depend on its symmetry. Moreover, it will be demonstrated later that the condition of closest packing of fibres parallel to fivefold axes is related to a particular icosahedral symmetry configuration.

Between the icosahedral point groups I_h and I (see *International Tables for Crystallography*, 1983), the number of symmetry operations (*i.e.* the group order) changes from 120 to 60 because of the loss of the centre of symmetry. With respect to the point group I, the additional elements of symmetry contained in the point group I_h have 15 mirror planes σ ($\sigma = C_2 * I$). As a consequence, transformations between these point groups might be envisaged. For instance, a toy model like that photographed in Fig. 2 can be built in order to prove such behaviour. This model has rigid faces of pentagonal and triangular shapes and articulations whose shape can vary from a pyramid of square basis up to a two-dimensional flattened configuration. For both of these particular shapes of articulation (corresponding to two polyhedra, one with pentagonal, square and triangular faces and the other the icosidodecahedron), the point-group symmetry is I_h and in between these limits the point group is I. Note that such a toy model exhibits a particular behaviour of retraction and rotations around five-, three- and twofold axes under compression, which might also be related to an elastic deformation of an icosahedral packing of fibres of point-group symmetry I under a compressing strain. Besides, as the point group I is noncentrosymmetric, enantiomorphic forms can be generated through clockwise and anticlockwise rotations, respectively, around the fivefold axes.

3. Experimental construction

3.1. *First step*

The beginning of the construction of the model shown in Fig. 1(*a*) is similar to that used by Parkhouse & Kelly (1998) to explain the relative positions of embryonic markers of two-dimensional Penrose tilings normal to the fivefold icosahedral axes. As shown in Figs. 3(*a*), (*b*), (*c*) and (*d*), a set of 30 fibres of circular cross section run normal to the faces of a dodecahedron. The first three perspective views (Figs. 3*a*, *b* and *c*) are related to the same set of 30 fibres successively oriented along icosahedral axes of fivefold, threefold and twofold symmetry. Each subset of five parallel fibres defines at its intersection with a pentagonal face of the dodecahedron the vertices of a pentagon whose orientation is at 18° rotation clockwise from the vertices of the pentagonal face. An enantiomorphic arrangement would be obtained for an 18° anticlockwise rotation. Both these enantiomorphic packing orientations satisfy the closest-packing condition that we shall demonstrate later (*cf.* §4.1). For this first packing step, each fibre is in contact with four other fibres. For a rotation different from 18° but not equal to zero, each fibre would only be in contact with three other fibres and the size of the arrangement would be largest for the same fibre diameter. Without rotation (*i.e.* for the same orientation of both pentagons), the packing is impossible as fibre axes of different fivefold axes run through common points. In fact, it is impossible to construct a packing of fibres of which the point-group symmetry would be I_h, even if the subsets of fibres parallel to each fivefold icosahedral axis are replaced by decagonal arrangements of ten fibres.

3.2. *Following steps: experimental procedure*

In order to build a packing of fibres with icosahedral symmetry, Parkhouse & Kelly (1998) considered the particular case where fibres would be normal and passing through all the nodes of a two-dimensional Penrose pattern (Penrose patterns are recapitulated in §4.3). However, we shall show in the following that this is not possible.

Our procedure was first experimental and the result was then described and demonstrated in hyperspace in order to determine unambiguously any fibre position using a method of cut and projection applied to this hyperspace description.

As also noticed by Parkhouse & Kelly (1998), an icosahedral fibre packing reduces to a two-dimensional problem. In that case, there is a geometrical characteristic to note from the fivefold projection of the first set of 30 fibres (Fig. 3*d*) which will be useful for going on to the packing of other fibres. It turns out that, to the subset of five fibres that are normal to the plane of the figure, it corresponds through a rotation of $(2k+1)\pi/10$ (with k integer) a set of fivefold intersections defined by the projection of the other fibres (and reciprocally). As a consequence, when adding new fibres, the corresponding new fivefold intersections can overlap neither with previous intersections nor with a fibre normal to the plane of the figure (if not, fibres would be passing through the bulk of other fibres, which is excluded).

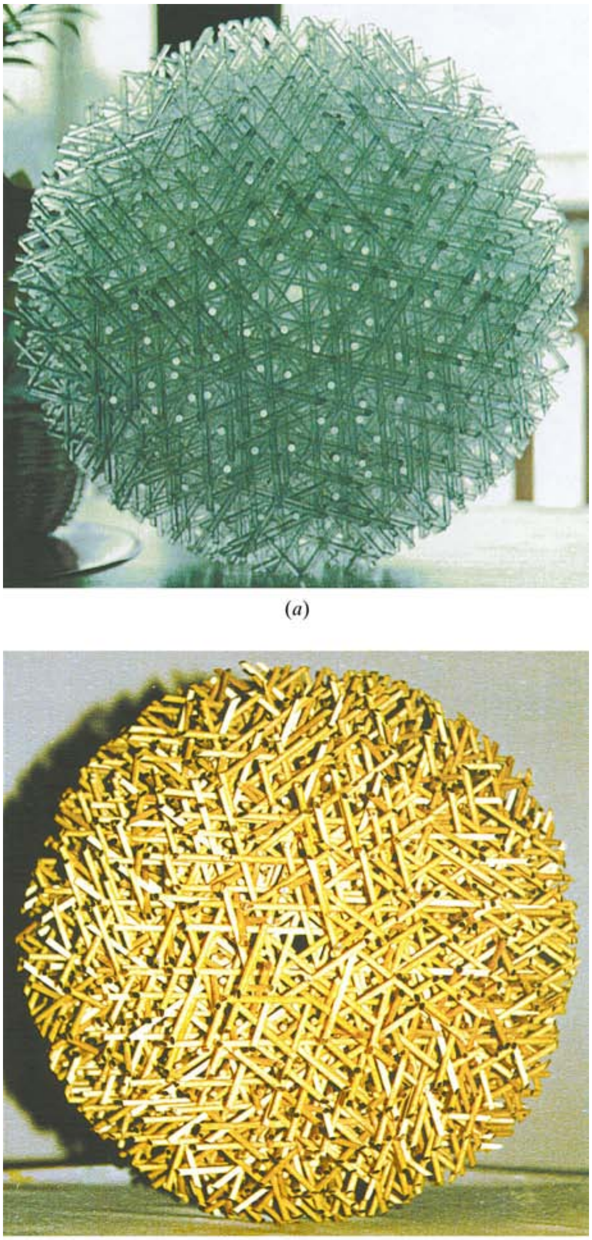
Therefore, from the projection shown in Fig. 3(*d*), one has to thread new fibres either through holes of the same shape as those related to the first subset of fibres normal to the projection or at the external border, keeping in mind that for each new fibre normal to the plane of the figure the corresponding fivefold intersections, related to $(2k+1)\pi/10$ rotations, must come inside holes. In this way, one easily finds where another subset of five fibres and five related intersections of fivefold symmetry can be placed (Figs. 3*e* and *f*). The same experimental procedure applies for all the further steps of this packing (Figs. 3*g* and *h*). As it will be proved that there is only one configuration for such a closest fibre packing (*cf.* §4.1), it is implied that the construction does not require the respect of a strict ordered sequence as a function of a regular increase in the radius of the fivefold projected pattern. The only requirement is that any new fibre must be threaded such that it is in contact with previous fibres.

3.3. *2D pattern and hole shapes*

As a result of this experimental construction, and although it is limited in the space, one can analyse the characteristics of the 2D pattern of fivefold symmetry and make a comparison with a Penrose pattern. In Fig. 3, two groups of fibres are distinguished by two colours because they define 2D patterns related by a τ inflation and a rotation of $(2k+1)\pi/5$ (k integer). τ is the golden mean [$\tau = (1 + 5^{1/2})/2$]. The fivefold projection of the first group is shown in Fig. 4(a) and its fibre subset normal to the projection plane is shown in Fig. 4(b) where part of a tiling constituted of pentagons, stars, large and thin lozenges, banana- and boat-shaped polygons is defined from line segments of the same length. Note that a complete tiling can be generated through a mirror operation and $k\pi/5$ rotations (k integer). A characteristic of this group is that it can be realized in practice as the fibres are in contact with other fibres. In addition to this first group, another group of fibres, distinguished by a different colour, has to be threaded in order to obtain the closest packing of fibres of the same cross section and only parallel to the fivefold axes (Fig. 4c). From its related fibre subset, normal to the projection plane, one defines a tiling that differs from the previous one only through a τ inflation and a rotation of $(2k + 1)\pi/5$ (k integer) (Fig. 4d).

The tiling shown in Fig. 4(b) can be compared with that shown in Fig. 5, only tiled with pentagons, thin lozenges, stars and boat-shaped polygons and which corresponds to a part of a Penrose tiling. In the case of this Penrose tiling, there are firstly five boat-shaped polygons around the centre and five stars further away. As this is the opposite in the case of the tiling shown in Fig. 4(b), one fibre should be removed from the star and one fibre should be added to the boat-shaped polygon in order to obtain the Penrose tiling configuration. However, if the removal of a fibre is possible, an addition is not. Therefore, such a fibre packing does not obey the Penrose tiling rules, making the construction problem more complicated than expected. In effect, with the specified Penrose tiling rules, it would be easy to extend in a deterministic way the construction of the fibre packing to any size, while, in the present case, we have not found any other solution than the one presented before which implies several tries for finding a new fibre position. This problem remains open.

There is a metrical relation between the line segment length (L) of the tiling shown in Fig. 4(b) and the fibre radius (R). Such a relation can simply be deduced from one of the hole shapes containing fibres normal to the projection plane (Fig. 6). From Fig. 6(a), one finds that $L = 4\tau(\tau + 2)^{1/2}R$. The other hole shapes containing fibres of the first packing group are shown in Figs. 6(b) and (c) and those related to both groups in Figs. 6(d), (e) and (f). From these figures, one can also deduce that fibres of the same decagonal cross section could be used in order to obtain the maximal volume fraction of reinforcement.



(a)

(b)

Fig. 1. Photographs of two different models of icosahedral packing of fibres: (a) model built with Plexiglas fibres whose axes are parallel to the six fivefold icosahedral axes; (b) model built with wood fibres whose axes are parallel to the six fivefold and ten threefold icosahedral axes.

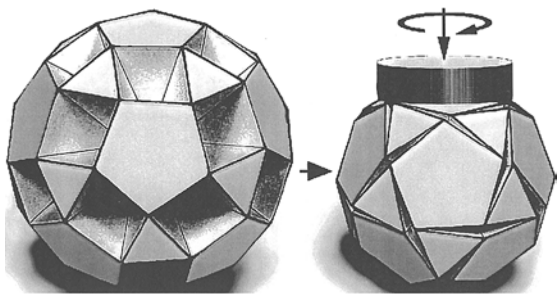
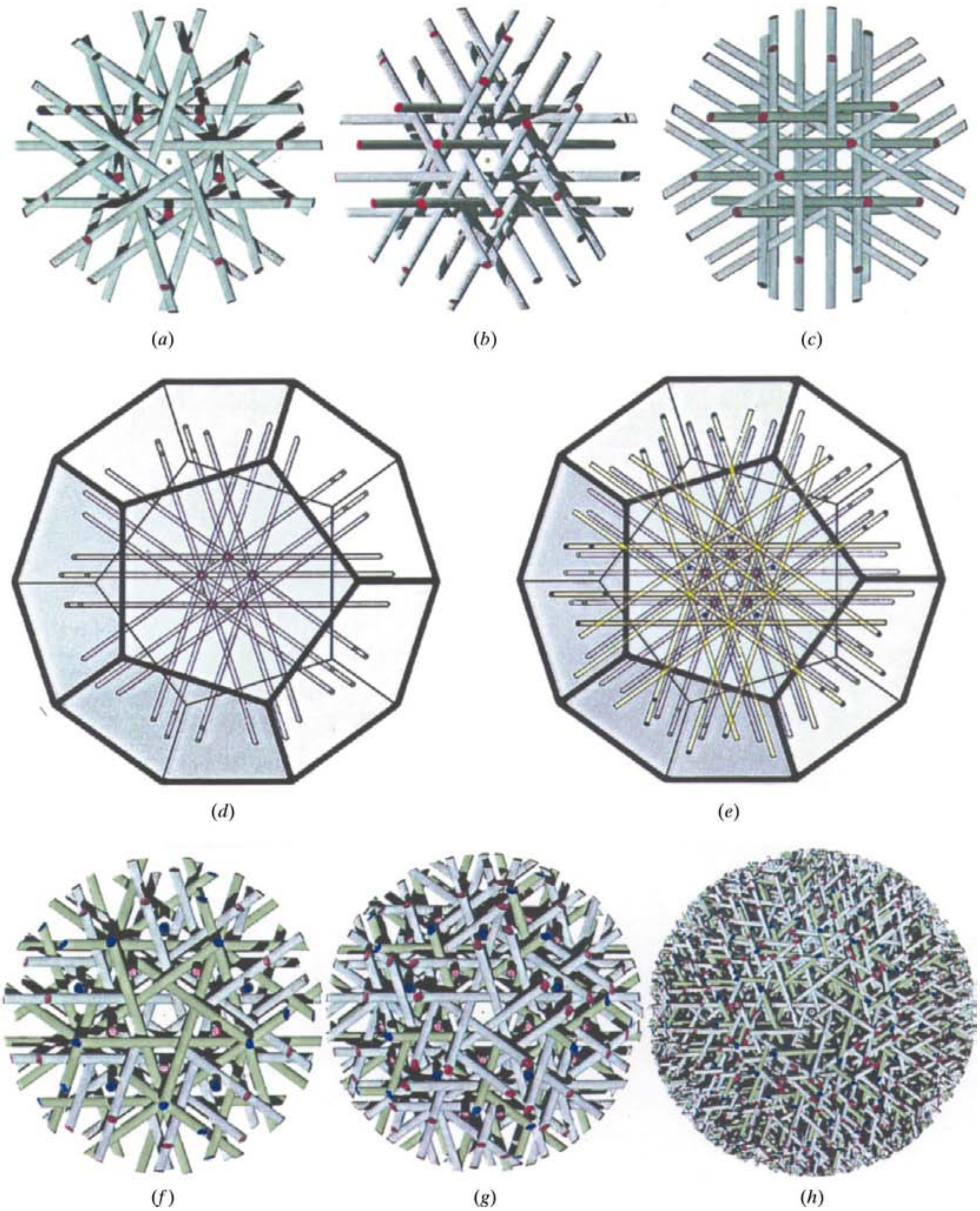


Fig. 2. The I_h to I to I_h icosahedral point-group transformation obtained from a toy model.

4. Mathematical construction

4.1. *The 5D construction*

We develop now the theoretical method that leads to and justifies the constructions presented above. As mentioned earlier, the set of fibres is completely specified by the subset of fibres that are parallel to a given fivefold axis. These fibres can be fixed by their intersections with the orthogonal plane running through the origin. The 3D description is thus reduced to a 2D problem that we shall handle in the general 5D framework of the Penrose tilings. Of course, this leaves out



(*a*) (*b*) (*c*) (*d*) (*e*) (*f*) (*g*) (*h*)

Fig. 3. Steps for constructing the tightest icosahedral packing with fibres of the same circular cross section and whose axes are only parallel to the six fivefold icosahedral axes: (*a*)–(*c*) configuration of the first step viewed in perspective and successively along fivefold, threefold and twofold icosahedral directions; (*d*), (*e*) first and second building steps showing that fibres can be threaded through holes drilled on the pentagonal facets of a dodecahedron; (*f*)–(*h*) perspective views of the second, third and *n*th steps. The two colours pink and yellow or grey and green are related to two subgroups of fibres as explained in the text.

possible structures that cannot be obtained by a cut-and-project method.

We assume that the x_3 axis is a fivefold axis of the icosahedral group and that the x_1 and x_2 axes are parallel and perpendicular to a twofold axis, respectively. The position of the icosahedral group is completely specified in Fig. 7 where the 15 twofold axes point to the mid edges of a dodecahedron. The black dot at point y represents a possible intersection of a vertical fibre with the (x_1, x_2) plane. The x_3 fivefold axis and the horizontal twofold axes generate nine more vertical fibres represented by grey dots, yielding a set of ten vertical fibres (this number reduces to five if the initial fibre is invariant with respect to a horizontal twofold axis). The icosahedral symmetry generates five copies of these ten vertical fibres, each set being parallel to a fivefold axis. Fig. 7 represents the projection of one set of ten oblique fibres into the horizontal plane. Thanks to the fivefold symmetry of the construction, we see that the non-intersection of fibres of different orbits can be handled in the following way: If two vertical fibres are located at $x = (x_1, x_2)$ (white dot) and $y = (y_1, y_2)$ (black dot), we just have to check that the line parallel to the x_2 axis running at $-y$ is not too close to the point x. The minimal distance between points and lines (to be specified below) dictates the largest possible diameter of disjoint fibres. Our aim is therefore to find the densest packing of fibres once a minimal distance (or diameter) is chosen.

We first consider five fibres based on a pentagon with a free rotation with respect to the x_3 fivefold axis of the icosahedral group. The construction of Fig. 7 shows that the largest distance between points (vertical fibres) and lines (projections of oblique fibres) is obtained when a vertex of the pentagon lies on the x_1 axis. This distance actually shrinks to 0 if the pentagon is parallel to the upper face of the dodecahedron. This fixes the relative orientation of $\pm 18^\circ$ between the icosahedral group (given by the dodecahedron) and the pentagonal basis indexing the positions of the fibres in the (x_1, x_2) plane. The minimal allowed distance is then $\tau^{-2}/2$ for a pentagon of unit radius and the corresponding radius of the fibres is therefore $\tau^{-2}/4$ as can be deduced from the discussion in §3.3 [*i.e.* where such a type of relation is expressed with respect to the edge length of the tiling shown in Fig. 4(*b*)].

Penrose tilings and related quasiperiodic fivefold patterns can be obtained by the cut-and-projection method from a 5D space. A point X of the 5D lattice $\mathbf{Z}^5$ has integer coordinates $(m_i, i = 1, \ldots, 5)$ with respect to the standard basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}$ and the two projections are denoted $\pi(X) = x = (x_1, x_2)$ in the 2D parallel space and $\pi'(X) = x' = (x'_1, x'_2, x'_3)$ in the 3D perpendicular space. The units are chosen in order that $e_k = \pi(\varepsilon_k)$ is a unit vector and that $e'_k = \pi'(\varepsilon_k)$ has a unit projection in the (x'_1, x'_2) plane of perpendicular space and unit projection on the x'_3 axis. Notice that e_1 falls on the x_1 axis and that the pentagon $\{e_1, \ldots, e_5\}$ is turned by 18° with respect to the upper facet of the dodecahedron.

The 3D perpendicular projection of the 5D lattice falls on planar layers L_n defined by $x'_3 = n$ where n is an integer: if $X = (m_i)$ then x' falls in the layer L_μ where $\mu = \sum m_i$. Since the diagonal $(1, 1, 1, 1, 1)$ has a null projection in the 2D parallel plane, we have merely to specify windows $W_0, \ldots, W_4$ in the five layers in $L_0, \ldots, L_4$. In order to ensure the fivefold symmetry of the construction, we shall assume that these windows have pentagonal symmetry and are centred on the x'_3 axis. The positions $x = (x_1, x_2)$ of vertical fibres will be given by a cut-and-project algorithm: if $X = (x, x')$ is a 5D lattice point, the position x is selected if x' falls in the interior of one of the windows W_μ, where $\mu = \sum m_i$ takes on five possible values $0, \ldots, 4$. Similarly, another point $Y = (n_i) = (y, y')$, selected by window W_ν with $\nu = \sum n_i$, yields another fibre located at y. As discussed above, we have to check that x is not too close to the line parallel to the x_2 axis running at point $-y$. This amounts to estimating $x_1 + y_1$, which is the coordinate z_1 of the 5D lattice point $Z = X + Y = (z, z')$. The coordinates x_1 and x'_1 actually depend only on two integers p and p' related to X. If we map the 5D lattice point X onto (p, p'), with $p = m_3 + m_4 - m_2 - m_5$ and $p' = 2m_1 - m_2 - m_5$, then we have $x_1 = (p\tau + p')/2$ and $x'_1 = (p + p' - p\tau)/2$. Now, the point is that we can interpret x_1 and x'_1 as the 'parallel' and 'perpendicular' projections of the 2D lattice point $\alpha = s(p, p')$ on the lines of slope τ^{-1} and $-\tau$ in the 2D plane (x_1, x'_1). The point α belongs to the square lattice $s\mathbf{Z}^2$, where the scaling factor s is equal to $(\tau + 2)^{1/2}/2$. We have similar mappings of Y onto $\beta = s(q, q')$ and of $Z = X + Y$ onto $\gamma = s(r, r')$ with $\gamma = \alpha + \beta$. This mapping from $\mathbf{Z}^5$ to $\mathbf{Z}^2$ has the further property $3p + p' = 2\sum m_i \pmod 5$ so that all 5D lattice points corresponding to a given layer L_μ in perpendicular space are mapped onto the 2D sublattice $3p + p' = 2\mu \pmod 5$, which is of index 5.

The condition $|x_1 + y_1| = |r\tau + r'|/2 \geq \tau^{-2}/2$ must be fulfilled for all selected points x' in W_μ and y' in W_ν. The minimal distance $\tau^{-2}/2 = (-\tau + 2)/2$ specified above corresponds to the projection of the points $P = s(-1, 2)$ and $P' = s(1, -2)$ which are therefore allowed sums $\alpha + \beta$. On the contrary, any lattice point γ in $s\mathbf{Z}^2$ with a smaller parallel component should not be obtained as a sum of two selected points α and β. In particular, the points $Q = s(2, -3)$, $Q' = -Q$, $R = s(3, -5)$ and $R' = -R$ must be avoided.

These constraints imply maximal sizes for the different windows. We have first considered self constraints for each window W_μ. W_0 must be empty and the best solutions in other layers are given by pentagonal windows: W_1 is a pentagon of radius $\rho_1 = (2/\tau)\tau_4/4 = \tau^3/2$, W_2 is a pentagon of radius $\rho_2 = (2/\tau)\tau^3/4 = \tau^2/2$ (see Fig. 8), $W_3 = -W_2$ and $W_4 = -W_1$. Then we have considered the constraints between pairs of different windows. A careful analysis of these conditions leads to the following conclusion: the highest density is obtained with the two windows W_1 in layer L_1 and W_2 in layer L_2 (or W_3 and W_4 by symmetry). Other combinations of two windows imply shrinking at least one of them and therefore lowering the density of fibres.

The proof of the above results is a bit technical. We just outline here how the maximal size of W_1 is obtained. If X and Y fall in the layer L_1 then $Z = X + Y$ falls in L_2 and the 2D lattice points $\alpha = s(p, p')$, $\beta = s(q, q')$ and $\gamma = s(r, r')$ satisfy $3p + p' = 2\,(\mathrm{mod}\,5)$, $3q + q' = 2\,(\mathrm{mod}\,5)$ and $3r + r' = 4\,(\mathrm{mod}\,5)$. The point $\gamma = s(3, -5) = R$ must be avoided. Since for this point we have $z'_1 = x'_1 + y'_1 = -\tau^4/2$, we must have $x'_1 > -\tau^4/4$ and $y'_1 > -\tau^4/4$ so that the window W_1 must be bounded on the left at $-\tau^4/4$. Therefore the largest W_1 window is the pentagon of radius $\rho_1 = (2/\tau)\tau^4/4 = \tau^3/2$.

We finally notice that the sets of positions selected by window W_2 can be deduced from the set selected by W_1 by a scaling by $-\tau$ in the (x_1, x_2) plane (see also Fig. 4). This follows from the property of the inflation mapping of the 5D cut-and-project construction.

4.2. *Density of the packing*

We now compute the density of fibres obtained with these windows. The density of points in the (x_1, x_2) plane is proportional to the total area of the windows. The number of points selected by W_1 and lying in a large box B of the (x_1, x_2) plane is close to $|B||W_1|$ multiplied by the density of lattice of points $X = (m_i)$ such that $\sum m_i = 1$. A careful computation shows that the density of points is $n(W_1) = 4|W_1|/(25 \times 5^{1/2})$ and we get a similar expression for the window W_2. This can be checked by considering the auxiliary 4D lattice spanned by $a_k - a_{k+1}$ $(k = 1, \ldots, 4)$, where

$$\begin{aligned} a_k &= (e_{k,1}, e_{k,2}, e'_{k,1}, e'_{k,2}) \\ &= (\cos(2k\pi/5), \sin(2k\pi/5), \cos(4k\pi/5), \sin(4k\pi/5)), \end{aligned}$$

the unit cell of which has volume $5^{1/2} \times 25/4$. The total density of points is therefore $n = 4(|W_1| + |W_2|)/(25 \times 5^{1/2})$, which evaluates to $n \approx 1.054\,74$ with the two pentagons of radius $\rho_1 = \tau^3/2$ and $\rho_2 = \tau^2/2$. The relative volume filled by such a distribution of fibres parallel to the x_3 axis is $n\pi(\tau^{-2}/4)^2 = n\pi\tau^{-4}/16$ (the maximal radius of the fibres was evaluated to be $\tau^{-2}/4$). Now, taking into account the six orientations of fibres, we get a volume fraction equal to $6n\pi\tau^{-4}/16 \approx 0.181\,291$. This result is obtained with fibres of circular section. However, a slightly higher volume can be reached with fibres of decagonal section as can be checked in Fig. 6.

4.3. *Comparison with the Penrose tiling*

Roger Penrose discovered three famous aperiodic tilings with fivefold symmetries (Penrose, 1974, 1979; Grünbaum & Shephard, 1987). The first one was constructed with six tiles: a star, a boat, a rhomb and three pentagons. Subsequently, he found two closely related tilings with only two prototiles, either a kite and a dart or two rhombs. These tilings were originally built by an inflation method, starting from a seed and applying a decomposition rule. All of them can also be obtained by means of a cut-and-project method in the framework presented above (see for instance Janot, 1994). The structure proposed by Parkhouse & Kelly (1998) is based on the so-called P1 tiling with stars, boats, rhombs and pentagons. The set of vertices can be recovered from a unique window P_1 which is a decagon of radius $(\tau + 2)/2 \approx 1.809$ shown in Fig. 8, which selects points in the layer L_1 of perpendicular space (*i.e.* points of $\mathbf{Z}^5$ such that $\sum m_i = 1$). Although the surfaces of P_1 and W_1 are almost equal ($|P_1| \approx 10.6331$ and $|W_1| \approx 10.6663$), the corresponding structures are slightly different as mentioned in §3.2. Furthermore, there is no local rule to transform one structure into the other. The decagon P_1 does not satisfy the conditions for non-intersection. It follows that it is impossible to avoid intersections of fibres of equal diameter $\tau^{-2}/2$ based on the P1 Penrose tiling. However, these intersections could be removed if some fibres were given a smaller diameter. Besides, our method yields a larger number of fibres since the window W_2 also contributes to the construction by adding a fraction $|W_2|/|W_1| = 2 - \tau$ of fibres.

5. Discussion

Our results show that only the cut-and-projection method applied to a 5D hyperspace yields a deterministic way for finding all the fibre positions in an icosahedral packing in which the fibre axes are parallel to the fivefold axes. But, as for brain-teaser games, one might wonder if simpler 2D building rules than those previously proposed could be deduced from the 2D pattern obtained by cut and projection, for instance building rules based on an inflation (or a deflation) procedure similar to that used for the 2D Penrose patterns (Grünbaum & Shephard, 1987). Note that such a problem might also be related to that of local rules for the growth of 3D quasicrystals.

Using similar but different building rules, we have also carried out the construction of packing of fibres whose axes are parallel to the ten threefold icosahedral axes. These results will be published in a forthcoming paper. Note that it is not necessary to try a construction where fibre axes would be parallel to the 15 twofold icosahedral axes because the following simple demonstration proves the case to be impossible:

For both point groups I and I_h, each fivefold axis is perpendicular to five twofold axes situated in the same plane. Such twofold axes are therefore related between them through the matrix operators C_5^n, $n = 1$ to 5. Let us suppose a fibre anywhere in the space but with its axis parallel to one of these twofold axis. Through the C_5^n, $n = 1$ to 4 transformations, four other fibres will have their axes in the same plane as the first one, then exhibiting intersecting points, which is physically excluded.

Besides, note that, for lines parallel to the 15 twofold icosahedral axes, the point-group symmetry is always I_h, which is in agreement with the symmetry property for an icosahedral packing that its point group is never I_h but always I.

Theoretically, the elasticity properties of an icosahedral packing of fibres are not so simple to predict as they might depend strongly on nonlinear terms. Experiments permitting a comparison of elasticity properties of composites with either periodic or quasiperiodic fibrous reinforcement should be quite useful in that case. Nevertheless, as such properties should improve as both the number of different fibre directions and the fibre volume fraction increase, an interesting case to consider is when fibres are threaded parallel to the six fivefold and ten threefold icosahedral axes. Although such a type of fibre packing has been built (Fig. 1*b*), we have not yet established its corresponding mathematical construction from cut and projection from a 6D space. Experimentally, the construction of this model was carried out as follows:

Considering both groups of fibres only parallel to fivefold icosahedral axes (Fig. 4), the τ-inflated group

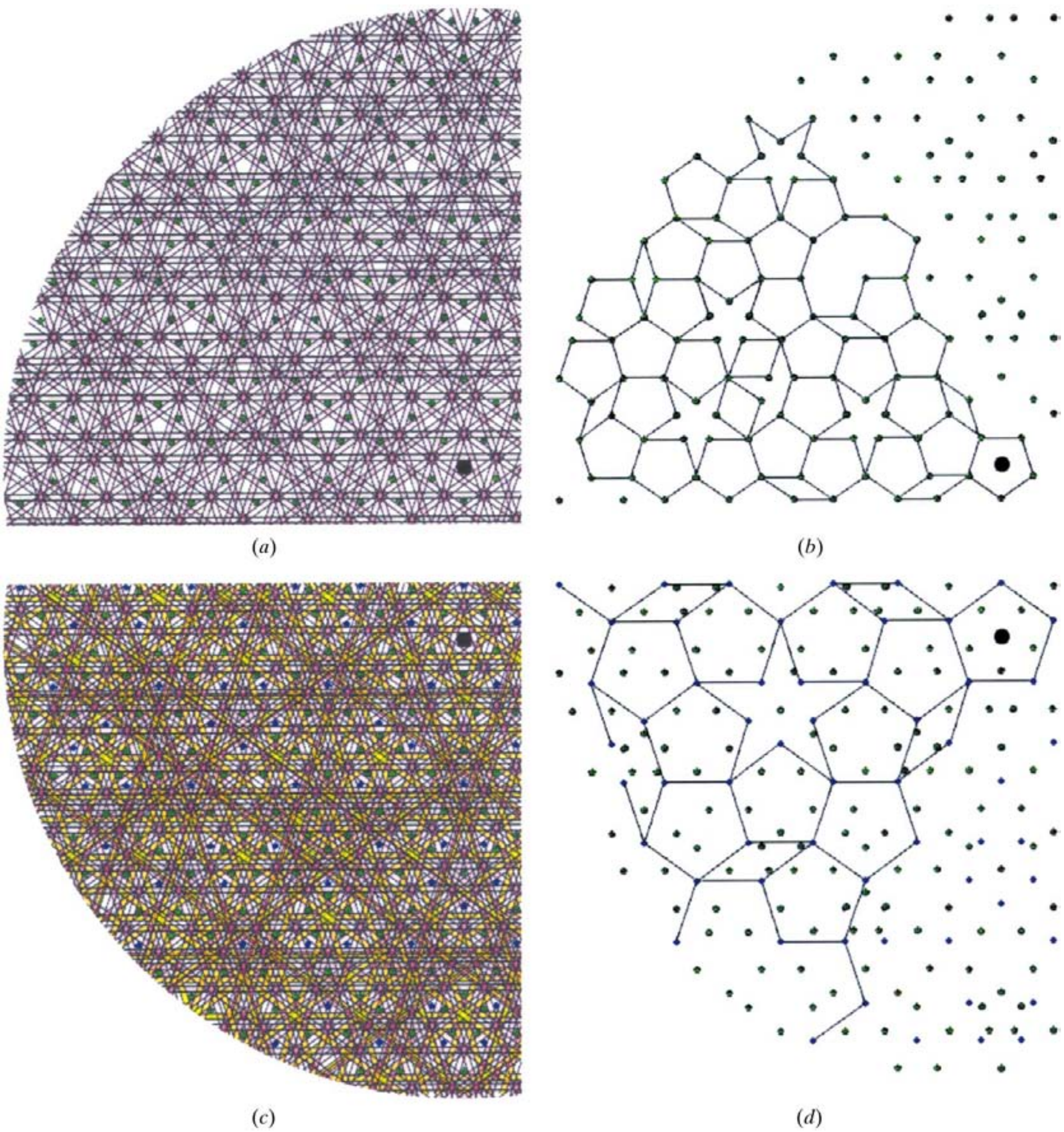


Fig. 4. Fivefold projections and corresponding 2D patterns related to both subgroups of fibres: (*a*) and (*b*) for the first subgroup where all the fibres are touching other fibres; (*c*) and (*d*) for both the previous subgroup and the τ-inflated subgroup.

has been removed. In that case, identical patterns of holes were observed in the planes perpendicular to the threefold icosahedral axes. We have found that such a hole pattern can be superimposed through a scaling factor onto the 2D pattern obtained when fibres are only threaded parallel to the threefold icosahedral axes. Therefore, the first group of fibres parallel to fivefold axes could be associated with the packing of fibres parallel to the threefold icosahedral axes but using two different diameters of fibre.

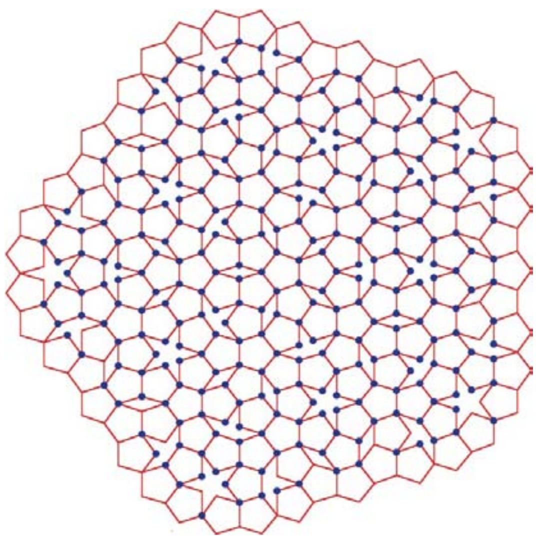


Fig. 5. The P1 Penrose tiling with pentagons, rhombs, stars and boats built by the cut-and-project method with a decagonal window in layer L_1 of perpendicular space.

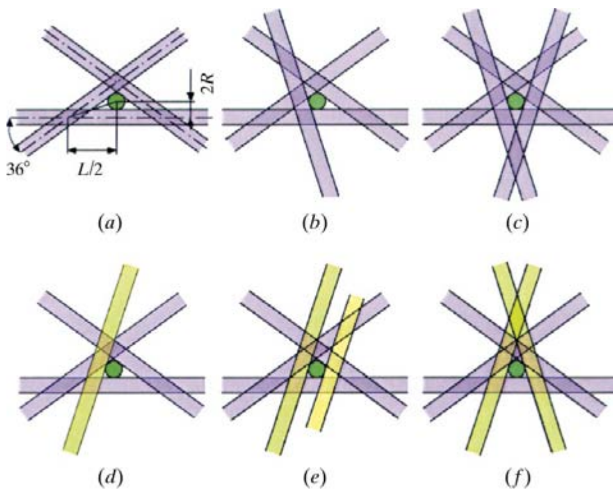


Fig. 6. Shapes of the holes through which fibres normal to the figure plane are threaded: (*a*)–(*c*) for the first subgroup of fibres shown in Fig. 4(*a*); (*d*)–(*f*) for both the subgroups shown in Fig. 4(*c*). A relation between R (the fibre radius) and L [the edge length of the tiling shown in Fig. 4(*b*)] can be deduced from (*a*).

Finally, besides mechanical properties, the photonic band-gap structure of icosahedral packing made with light transparent fibres could be interesting to examine as such a characteristic is similar to that of the electronic band-gap structure. Yablonovitch (1987) proved a few years ago that the theory on electronic structure of

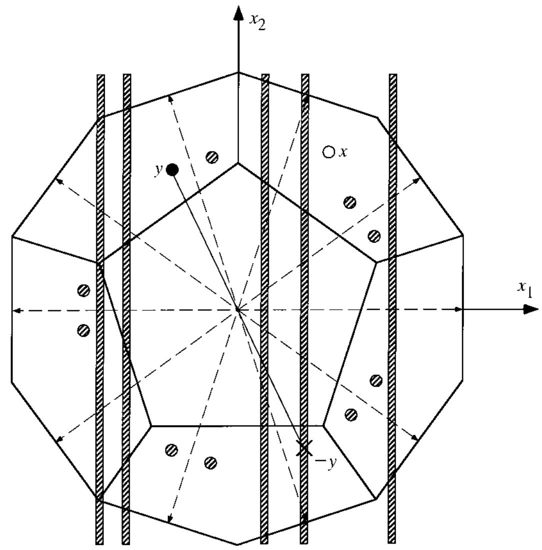


Fig. 7. The dodecahedron sets the orientation of the icoshadral group. Twofold axes lying in the horizontal plane (x_1, x_2) are represented by dotted lines. The white (x) and black (y) dots represent possible intersections of vertical fibres with the horizontal plane. The orbit of the fibre running at y yields nine more vertical fibres (grey dots). Ten other oblique fibres of the same orbit are represented by their projections. Their distances with respect to the fibre at x are of the form $x_1 + y_1$.

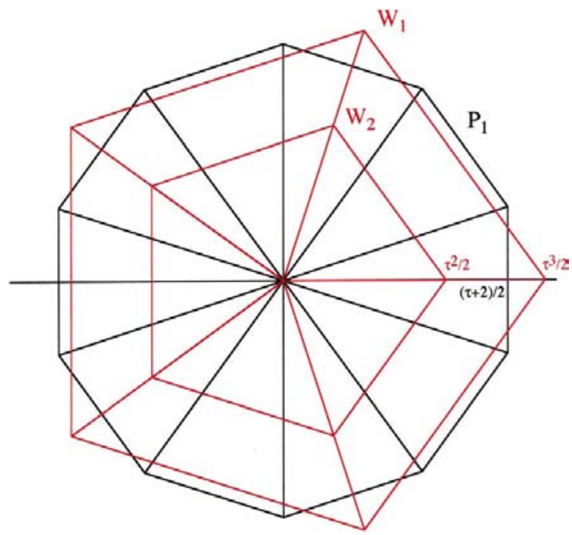


Fig. 8. The pentagonal windows W_1 in layer L_1 and W_2 in layer L_2 give the highest density of fibres. By comparison, the decagonal window P_1 in layer L_1 yields the Penrose tiling.

crystals applied also to the photonic structure of hand-made periodic light-transparent objects. In the case of intermetallic quasicrystals, a few electronic properties, such as very high resistivity, have been interpreted to be linked to an electronic pseudo-gap at the Fermi level situated on a first pseudo-Brillouin zone. But as a formal theory has not yet been developed in the case of the 3D quasiperiodic matter (or for 2D), several studies have been performed in order to determine experimentally the shape of this pseudo-gap. Therefore, it might be interesting to verify if an equivalent photonic pseudo-gap exists for a quasiperiodic fibre packing and, if it is true, what is its shape in order to make comparisons with conjectures proposed in the literature.

The authors thank Didier Landru for his help in a few drawings and Dr Robert Bellissent for his encouragement.

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